

MAXIMUM PRINCIPLE FOR SEMI-ELLIPTIC TRACE OPERATORS AND GEOMETRIC APPLICATIONS

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ABSTRACT. Based on ideas of L. Alías, D. Impera and M. Rigoli developed in [13], we present a fairly general weak/Omori-Yau maximum principle for trace operators. We apply this version of maximum principle to generalize several higher order mean curvature estimates and to give an extension of Alias-Impera-Rigoli Slice Theorem of [13, Thm. 16 & 21], see Theorems 5, 6.

1. INTRODUCTION

The theory of minimal and constant mean curvature hypersurfaces of product spaces $N \times \mathbb{R}$, where N is a complete Riemannian manifold, has been developed into a rich theory [30], [40], [41] yielding a wealth of examples and results, see for instance [1], [3], [4], [5], [6], [7], [11], [15], [17], [20], [21] [22], [24], [27], [32] [33], [34] and the references therein.

Recently, the theory minimal and constant mean curvature hypersurfaces started to be developed in more general spaces, as in the work of S. Montiel [31] and Alías-Dajczer [9], [10] where they studied constant mean curvature hypersurfaces in warped product manifolds $M^{n+1} = \mathbb{R} \times_{\rho} \mathbb{P}^n$, where \mathbb{P}^n is a complete Riemannian manifold and $\rho: \mathbb{R} \rightarrow \mathbb{R}_+$ is a smooth warping function. Those studies were further extended by Alías, Dajczer and Rigoli [12] and Alías, Impera and Rigoli [13] to include constant higher order mean curvature hypersurfaces in warped product manifolds and in general setting by Albanese, Alías and Rigoli [2].

In this paper we give a small contribution to the theory proving appropriate extensions the results of [13]. We start in Section 2 presenting general conditions for the validity of the weak maximum principle for a fairly general class of semi-elliptic trace operators 4, see Theorem 1. These operators and versions of Omori-Yau maximum principle were considered by Alias, Impera and Rigoli [13] and by Hong and Sung [25] under slightly more restrictive conditions. Then, we derive few geometric conditions on a manifold that guarantee that Theorem 1 applies, see Corollary 1 and Theorem 2. In section 4 we consider the L_r operators and prove several higher order mean curvature estimates for hypersurfaces immersed into warped product spaces $M^{n+1} = \mathbb{R} \times_{\rho} \mathbb{P}^n$, see Theorems 3, 4. In section 5 we extend the Slice Theorem of Alías-Impera-Rigoli [13, Thms. 16 & 21], see Theorems 5, 6.

2. MAXIMUM PRINCIPLE FOR TRACE OPERATORS

Following the terminology introduced in [37] we say that the Omori-Yau maximum principle for the Laplacian holds on M if for any given $u \in C^2(M)$ with $u^* = \sup_M u < +\infty$, there exists a

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sequence of points $x_k \in M$, depending on M and u , such that

$$(1) \quad \lim_{k \rightarrow +\infty} u(x_k) = u^*, \quad |\text{grad } u|(x_k) < \frac{1}{k}, \quad \Delta u(x_k) < \frac{1}{k}.$$

Likewise, the Omori-Yau maximum principle *for the Hessian* is said to hold on M if for any given $u \in C^2(M)$ with $u^* = \sup_M u < +\infty$, there exists a sequence of points $x_k \in M$, depending on M and on u , such that

$$(2) \quad \lim_{k \rightarrow +\infty} u(x_k) = u^*, \quad |\text{grad } u|(x_k) < \frac{1}{k}, \quad \text{Hess } u(x_k)(X, X) < \frac{1}{k} \cdot |X|^2,$$

for every $X \in T_{x_k} M$. Accordingly, the classical results of Omori [35] and Yau [42] can be stated saying that the Omori-Yau maximum principle for the Laplacian holds on Riemannian manifold with Ricci curvature bounded from below. The importance of the Omori-Yau maximum principle lies on its wide range of applications in geometry and analysis. Applications that goes from the generalized Schwarz lemma [43] to the study of the group of conformal diffeomorphism of a manifold [36], from curvature estimates on submanifolds [2], [6], [7] to Calabi conjectures on minimal hypersurfaces [26], [35]. The essence of the Omori-Yau maximum principle was captured by Pigola, Rigoli and Setti in Theorem 1.9 of [37] whose corollary is the following result: The Omori-Yau maximum principle holds on every Riemannian manifold M with Ricci curvature satisfying $\text{Ric}_M(\text{grad } \rho, \text{grad } \rho) \geq -C^2 G(\rho)$ where ρ is the distance function on M to a point, C is positive constant and $G : [0, +\infty) \rightarrow [0, \infty)$ is a smooth function satisfying $G(0) > 0$, $G'(t) \geq 0$, $\int_0^{+\infty} \frac{ds}{\sqrt{G(s)}} = +\infty$ and $\limsup_{t \rightarrow +\infty} \frac{tG(\sqrt{t})}{G(t)} < +\infty$.

In most applications of the Omori-Yau maximum principle, the condition $|\text{grad } u|(x_k) < 1/k$ is redundant, which led to the following definition.

Definition 1 (Pigola-Rigoli-Setti). *The weak maximum principle holds on a Riemannian manifold M if for every $u \in C^2(M)$ with $u^* = \sup_M u < +\infty$, there exists a sequence of points $x_k \in M$, such that*

$$(3) \quad \lim_{k \rightarrow +\infty} u(x_k) = u^*, \quad \Delta u(x_k) < \frac{1}{k}.$$

This apparently simple minded definition proved to be surprisingly deep. For instance, it has been proven in [36], that the weak maximum principle for the Laplacian is equivalent to the stochastic completeness of the diffusion process associated to Δ .

The Omori-Yau/weak maximum principle can be considered for differential elliptic operators other than the Laplacian, like the ϕ -Laplacian [38], the weighted Laplacian $\Delta_f = e^f \text{div}(e^{-f} \text{grad})$ [18], [29] and semi-elliptic trace operators $L = \text{Tr}(P \circ \text{hess})$ considered in [13], [28] and in [16], where $P : TM \rightarrow TM$ is a positive semi-definite symmetric tensor on TM and for each $u \in C^2(M)$, $\text{hess } u : TM \rightarrow TM$ is a symmetric operator defined by $\text{hess } u(X) = \nabla_X \text{grad } u$ for every $X \in TM$. Here ∇ be the Levi-Civita connection of M . In this paper we are going to consider the trace operator

$$(4) \quad Lu = \text{Tr}(P \circ \text{hess } u) + \langle V, \text{grad } u \rangle$$

with $\limsup_{x \rightarrow \infty} |V|(x) < +\infty$, and prove an Omori-Yau maximum principle in the same spirit of Theorem 1.9 of [37]. Then we prove some geometric applications that extends those of [13]. In [25], K. Hong and C. Sung considered the same trace operator and proved an Omori-Yau maximum principle but their proof required the stronger condition $\sup_M \text{Tr}(P) + \sup |V| < \infty$. It should be pointed out that Albanese, Alias and Rigoli in [2] recently proved an all general Omori-Yau maximum principle for trace operators of the form $Lu = \text{Tr}(P \circ \text{hess } u) + \text{div } P(\text{grad } u) + \langle V, \text{grad } u \rangle$.

However, the main purpose of this paper is to extend the geometric applications involving the operator (4) proved in [13] and for the sake of completeness we keep the proof of our version of the Omori-Yau maximum principle, besides it is very simple.

Our first result is the following extension of [13, Thm. 1].

Theorem 1. *Let $(M, \langle \cdot, \cdot \rangle)$ be a complete Riemannian manifold. Consider a semi-elliptic operator $L = \text{Tr}(P \circ \text{hess}(\cdot)) + \langle V, \text{grad}(\cdot) \rangle$, where $P : TM \rightarrow TM$ is a positive semi-definite symmetric tensor and V satisfies $\limsup_{x \rightarrow \infty} |V|(x) < +\infty$. Suppose that exists a non-negative function $\gamma \in C^2(M)$ satisfying:*

- i) $\gamma(x) \rightarrow +\infty$ as $x \rightarrow \infty$,
- ii) $\exists A > 0$ such that $|\nabla \gamma| \leq A\sqrt{G(\gamma)} \cdot \left(\int_0^\gamma \frac{1}{\sqrt{G(s)}} ds + 1 \right)$ off a compact set,
- iii) $\exists B > 0$ such that $\text{Tr}(P \circ \text{hess} \gamma) \leq B\sqrt{G(\gamma)} \cdot \left(\int_0^\gamma \frac{ds}{\sqrt{G(s)}} + 1 \right)$ off a compact set.
- iv) Where $G : [0, \infty) \rightarrow [0, \infty)$ is such that: $G(0) > 0$ $G'(t) \geq 0$ and $G(t)^{-\frac{1}{2}} \notin L^1(+\infty)$.

Then given any function $u \in C^2(M)$ that satisfies

$$(5) \quad \lim_{x \rightarrow \infty} \frac{u(x)}{\varphi(\gamma(x))} = 0,$$

where

$$(6) \quad \varphi(t) = \ln \left(\int_0^t \frac{ds}{\sqrt{G(s)}} + 1 \right),$$

there exists a sequence $\{x_k\}_{k \in \mathbb{N}} \subset M$ satisfying:

$$(a) \quad |\nabla u|(x_k) < \frac{1}{j} \quad \text{and} \quad (b) \quad Lu(x_k) < \frac{1}{j}.$$

If instead of (5) we suppose that u is bounded above we have that

$$(c) \quad \lim_{k \rightarrow +\infty} u(x_k) = \sup u.$$

2.1. Proof of Theorem 1. The proof of Theorem 1 we will follow the same steps of the proof of the maximum principle of Omori-Yau in the case of the Laplacian and the f -Laplacian presented in [29]. Thus, we only prove the item (b). Consider the following family of functions

$$f_k(x) = u(x) - \varepsilon_k \varphi(\gamma(x)),$$

where φ is defined in (6) and $\varepsilon_k \rightarrow 0^+$ when $k \rightarrow +\infty$. Observe that the condition (5) implies that f_k reaches a local maximum, say at $x_k \in M$. Suppose that the sequence $\{x_k\}_{k \in \mathbb{N}}$ diverges, (leaves any compact subset of M) otherwise we have nothing to prove.

Using the fact that x_k be the point of maximum to f_k we infer that

$$(7) \quad 0 = \text{grad } u(x_k) - \varepsilon_k \varphi'(\gamma(x_k)) \text{grad } \gamma(x_k)$$

and for all $v \in T_k M$ we have

$$(8) \quad 0 \geq \text{Hess } u(x_k)(v, v) - \varepsilon_k [\varphi'(\gamma(x_k)) \text{Hess } \gamma(x_k)(v, v) - \varphi''(\gamma(x_k)) \langle \text{grad } \gamma(x_k), v \rangle^2].$$

Calculating φ' and φ'' we obtain

$$(9) \quad \varphi'(t) = \left\{ \left(\int_0^t \frac{ds}{\sqrt{G(s)}} + 1 \right) \sqrt{G(t)} \right\}^{-1}$$

and

$$(10) \quad \varphi''(t) = - \left\{ \sqrt{G(t)} \left(\int_0^t \frac{1}{\sqrt{G(s)}} ds + 1 \right) \right\}^{-2} \left\{ \frac{G'(t)}{2\sqrt{G(t)}} \left(\int_0^t \frac{1}{\sqrt{G(s)}} ds + 1 \right) + 1 \right\} \leq 0$$

Taking (9) into (7), we have by the hypothesis ii)

$$(11) \quad |\text{grad } u|(x_k) \leq \varepsilon_k |\varphi'(\gamma(x_k))| |\text{grad } \gamma|(x_k) \leq \varepsilon_k.$$

Taking (10) into (8), we get

$$(12) \quad \text{Hess } u(x_k)(v, v) \leq \varepsilon_k \varphi'(\gamma(x_k)) \text{Hess } \gamma(x_k)(v, v).$$

Choose a basis of eigenvectors $\{v_1, \dots, v_n\} \subset T_{x_k} M$ of $P(x_k)$, corresponding to the eigenvalues $\lambda_j(x_k) = \langle P(x_k)v_j, v_j \rangle \geq 0$, with $1 \leq j \leq n = \dim(M)$. Therefore, by the inequality (12) it follows that

$$\begin{aligned} \langle P(x_k) \text{hess } u(x_k)v_j, v_j \rangle &= \lambda_j(x_k) \text{Hess } u(x_k)(v_j, v_j) \\ &\leq \varepsilon_k \varphi'(\gamma(x_k)) \langle P(x_k) \text{hess } \gamma(x_k)v_j, v_j \rangle \end{aligned}$$

Applying the trace on both sides of the inequality above and using (9) with the hypothesis iii), we have

$$(13) \quad \text{Tr}(P \circ \text{hess } u(x_k)) \leq \varepsilon_k \varphi'(\gamma(x_k)) \text{Tr}(P \circ \text{hess } \gamma(x_k)).$$

Therefore, by the item (a) and that $\limsup_{x \rightarrow \infty} |V|(x) < +\infty$ we have

$$\begin{aligned} L u(x_k) &= \text{Tr}(P \circ \text{hess } u(x_k)) + \langle V(x_k), \text{grad } u(x_k) \rangle \\ &\leq \varepsilon_k \varphi'(\gamma(x_k)) \text{Tr}(P \circ \text{hess } \gamma(x_k)) + D |\text{grad } u(x_k)| \\ &\leq (D + 1) \varepsilon_k. \end{aligned}$$

This finish the proof of Theorem 1.

Remark 1. *The Theorem 1 can be proved substituting the conditions ii) and iii) by the apparently more general conditions:*

$$\begin{aligned} \text{i. } |\nabla \gamma| &\leq A \prod_{j=1}^{\ell} \left[\ln^{(j)} \left(\int_0^t \frac{ds}{\sqrt{G(s)}} + 1 \right) + 1 \right] \left(\int_0^t \frac{ds}{\sqrt{G(s)}} + 1 \right) \sqrt{G(t)} \text{ and} \\ \text{ii. } \text{Tr}(P \circ \text{hess } \gamma) &\leq B \prod_{j=1}^{\ell} \left[\ln^{(j)} \left(\int_0^t \frac{ds}{\sqrt{G(s)}} + 1 \right) + 1 \right] \left(\int_0^t \frac{ds}{\sqrt{G(s)}} + 1 \right) \sqrt{G(t)} \end{aligned}$$

respectively. Just consider in the proof $\varphi(t) = \log^{(\ell+1)} \left(\int_0^t \frac{ds}{\sqrt{G(s)}} + 1 \right)$.

Corollary 1. *Let (M, \langle, \rangle) be a complete, non-compact, Riemannian manifold with radial sectional curvature satisfying*

$$(14) \quad K_M \geq -B^2 \prod_{j=1}^{\ell} \left[\ln^{(j)} \left(\int_0^r \frac{ds}{\sqrt{G(s)}} + 1 \right) + 1 \right]^2 G(r), \text{ for } r(x) \gg 1,$$

where $G \in C^\infty([0, +\infty))$ is even at the origin and satisfies iv), $r(x) = \text{dist}_M(x_0, x)$ and $B \in \mathbb{R}$. Then, the Omori-Yau maximum principle for any semi-elliptic operator $L = \text{Tr}(P \circ \text{hess}(\cdot)) + \langle V, \text{grad}(\cdot) \rangle$ with $\text{Tr } P \leq \left(\int_0^r \frac{ds}{\sqrt{G(s)}} + 1 \right)$ and $\limsup_{x \rightarrow \infty} |V| < +\infty$ holds on M .

Proof. Following the same steps of the example [37, Example 1.13] one has that bound (14) implies

$$\text{Hess } r \leq D \prod_{j=1}^{\ell} \left[\ln^{(j)} \left(\int_0^r \frac{ds}{\sqrt{G(s)}} + 1 \right) + 1 \right] \sqrt{G(r)}$$

and this implies

$$\text{Tr}(P \circ \text{hess}) \leq D \prod_{j=1}^{\ell} \left[\ln^{(j)} \left(\int_0^r \frac{ds}{\sqrt{G(s)}} + 1 \right) + 1 \right] \left(\int_0^r \frac{ds}{\sqrt{G(s)}} + 1 \right) \sqrt{G(r)}.$$

Hence Theorem 1 applies. \square

3. IMMERSIONS INTO WARPED PRODUCTS

By $L^\ell \times_\rho P^n = N^{n+\ell}$ we denote the product manifold $L^\ell \times P^n$ endowed with the warped product metric $dL^2 + \rho^2(x) dN^2$, where L^ℓ and P^n are Riemannian manifolds and $\rho : L \rightarrow \mathbb{R}_+$ is a positive smooth function. We will need the following definition introduced in [8].

Definition 2. Let M be a Riemannian manifold. We say that $(G, \tilde{\gamma})$ is an Omori-Yau pair for the Hessian in M if $\tilde{\gamma} \in C^2(M)$ is proper and satisfies

$$|\text{grad } \tilde{\gamma}| \leq \sqrt{G(\tilde{\gamma})} \left(\int_0^{\tilde{\gamma}} \frac{ds}{\sqrt{G(s)}} + 1 \right) \quad (15)$$

$$\text{Hess } \tilde{\gamma} \leq \sqrt{G(\tilde{\gamma})} \left(\int_0^{\tilde{\gamma}} \frac{ds}{\sqrt{G(s)}} + 1 \right)$$

Remark 2. If a Riemannian manifold M has an Omori-Yau pair for the Hessian $(G, \tilde{\gamma})$ then the Omori-Yau maximum principle for the Hessian holds on M , see [8], [29].

The following result gives conditions for an isometric immersion $f : M^m \rightarrow L^\ell \times_\rho P^n = N^{n+\ell}$ into a warped product, where P^n carries an Omori-Yau pair $(G, \tilde{\gamma})$ for the Hessian, to carry an Omori-Yau pair.

Theorem 2. Let $f : M^m \rightarrow L^\ell \times_\rho P^n = N^{n+\ell}$ be an isometric immersion where P^n carries an Omori-Yau pair $(G, \tilde{\gamma})$ for the Hessian, $\rho \in C^\infty(L)$ is a positive function, such that $\inf \rho > 0$ and $\mathcal{H} = \text{grad } \log \rho$ satisfies

$$|\mathcal{H}|(\pi_L(f)) \leq \ln \left(\int_0^{\pi_L(f)} \frac{ds}{\sqrt{G(s)}} + 1 \right). \quad (16)$$

If f is proper on the first entry and

$$|\alpha| \leq \ln \left(\int_0^{\tilde{\gamma} \circ \pi_P(f)} \frac{ds}{\sqrt{G(s)}} + 1 \right), \quad (17)$$

where π_L, π_P are the projections on L^ℓ and P^n respectively, and α is the second fundamental form of the immersion f , then M^m has an Omori-Yau pair for any semi-elliptic operator

$$L = \text{Tr}(P \circ \text{hess}(\cdot)) + \langle V, \text{grad}(\cdot) \rangle \quad \text{with} \quad \text{Tr } P \leq \prod_{j=2}^k \left[\ln^{(j)} \left(\int_0^{\tilde{\gamma} \circ \pi_P(f)} \frac{ds}{\sqrt{G(s)}} + 1 \right) + 1 \right].$$

Proof. By abuse of language, we will denote by $\langle \cdot, \cdot \rangle_N, \langle \cdot, \cdot \rangle_L, \langle \cdot, \cdot \rangle_P$ and $\langle \cdot, \cdot \rangle_M$ the Riemannian metrics on $N^{n+\ell}, L^\ell, M^m$ and on P^n respectively. Let $\| \cdot \|_{N,L,M,P}$ be their respective norms. We will denote by X, Y , vector fields in TL and by W, Z vector fields in TP . Let ∇^N, ∇^P and ∇^L denote the Riemannian connections on N, P and L respectively. We need few lemmas to prove Corollary 2.

Lemma 1. *The proof of the following relations are straight forward.*

$$(18) \quad \nabla_X^N Y = \nabla_Y^L X, \quad \nabla_Z^N X = \nabla_X^N Z = X(\eta)Z, \quad \nabla_Z^N W = \nabla_Z^P W - \langle Z, W \rangle \text{grad}^L \eta,$$

where $\eta = \log \rho$.

Recall that P carries a Omori-Yau pair $(G, \tilde{\gamma})$. Letting $\pi_P: N^{\ell+n} \rightarrow P^n$ be the projection on the second factor we define $\beta: N \rightarrow \mathbb{R}$ by $\beta = \tilde{\gamma} \circ \pi_P \circ e$ by $\gamma: M \rightarrow \mathbb{R}$ by $\gamma = \beta \circ f$. It is clear that

$$(19) \quad \langle \text{grad } \beta, X \rangle_N = 0 \quad \text{and} \quad \langle \text{grad } \tilde{\gamma}, Z \rangle_P = \langle \text{grad } \beta, Z \rangle_N = \rho^2 \langle \text{grad } \beta, Z \rangle_P.$$

Thus $\text{grad } \beta = \frac{1}{\rho^2} \text{grad } \tilde{\gamma}$, $\|\text{grad } \tilde{\gamma}\|_N = \rho \|\text{grad } \tilde{\gamma}\|_P$ and

$$|\text{grad } \gamma|_M \leq \|\text{grad } \beta\|_N = \frac{1}{\rho^2} \|\text{grad } \tilde{\gamma}\|_N = \frac{1}{\rho} \|\text{grad } \tilde{\gamma}\|_P.$$

Moreover, for all $e \in TM$ we have that, (identifying $f_*e = e$),

$$(20) \quad \text{Hess}_M \gamma(e, e) = \text{Hess}_N \beta(e, e) + \langle \text{grad } \beta, \alpha(e, e) \rangle_N.$$

Here α is the second fundamental form of the immersion. Let us write $e = X + Z$ where $X \in TL$ and $Z \in TP$ and we have that $\text{Hess}_N(e, e) = \text{Hess}_N(X, X) + 2\text{Hess}_N(X, Z) + \text{Hess}_N(Z, Z)$ and

$$(21) \quad \text{Hess}_N \beta(X, X) = \langle \nabla_X^N \text{grad } \beta, X \rangle_N = \langle X(\eta) \text{grad } \beta, X \rangle_N = 0.$$

$$(22) \quad \text{Hess}_N \beta(X, Z) = \langle \nabla_X^N \text{grad } \beta, Z \rangle_N = X(\eta) \langle \text{grad } \beta, Z \rangle_N = \langle \text{grad } \eta, X \rangle_L \langle \text{grad } \tilde{\gamma}, Z \rangle_P.$$

$$(23) \quad \begin{aligned} \text{Hess}_N \beta(Z, Z) &= \langle \nabla_Z^N \text{grad } \beta, Z \rangle_N \\ &= \langle \nabla_Z^P \text{grad } \tilde{\gamma}, Z \rangle_N - \langle \text{grad } \beta, Z \rangle_N \langle \text{grad } \eta, Z \rangle_N \xrightarrow{0} \\ &= \frac{1}{\rho^2} \langle \nabla_Z^P \text{grad } \tilde{\gamma}, Z \rangle_N = \text{Hess}_P \tilde{\gamma}(Z, Z). \end{aligned}$$

Hence from (21, 22, 23) we have that

$$(24) \quad \text{Hess}_N \beta(e, e) = 2 \langle \text{grad } \eta, X \rangle_L \langle \text{grad } \tilde{\gamma}, Z \rangle_P + \text{Hess}_P \tilde{\gamma}(Z, Z).$$

Thus, from (20),

$$\begin{aligned}
 \text{Hess}_M \gamma(e, e) &= 2\langle \text{grad } \eta, X \rangle_L \langle \text{grad } \tilde{\gamma}, Z \rangle_P + \text{Hess}_P \tilde{\gamma}(Z, Z) + \langle \text{grad } \beta, \alpha(e, e) \rangle_N \\
 &= 2\langle \text{grad } \eta, X \rangle_L \langle \text{grad } \tilde{\gamma}, Z \rangle_P + \text{Hess}_P \tilde{\gamma}(Z, Z) + \frac{1}{\rho^2} \langle \text{grad } \tilde{\gamma}, \alpha(e, e) \rangle_N \\
 (25) \quad &\leq 2\|\mathcal{H}\|_L \cdot \|\text{grad } \tilde{\gamma}\|_P \cdot \|X\|_L \cdot \|Z\|_P + \text{Hess}_P \tilde{\gamma}(Z, Z) + \frac{1}{\rho^2} \langle \text{grad } \tilde{\gamma}, \alpha(e, e) \rangle_N \\
 &\leq \ln \left(\int_0^\gamma \frac{ds}{\sqrt{G(s)}} + 1 \right) \sqrt{G(\gamma)} \left(\int_0^\gamma \frac{ds}{\sqrt{G(s)}} + 1 \right) \left(3 + \frac{1}{\rho} \right) |e|^2
 \end{aligned}$$

off a compact set, since $\ln \left(\int_0^\gamma \frac{ds}{\sqrt{G(s)}} + 1 \right) > 1$ there.

Let $x \in M$ and choose a basis $\{e_1, \dots, e_n\}$ for $T_x M$ formed by eigenvectors of $P(x)$ with eigenvalues $\lambda_j(x) = \langle P(x)e_j, e_j \rangle \geq 0$. Since

$$L\gamma(x) = \sum_{i=1}^n \langle P(x)(\text{hess } \gamma(x)(e_i)), e_i \rangle = \sum_{i=1}^n \langle \text{hess } \gamma(x)(e_i), P(x)(e_i) \rangle = \sum_{i=1}^n \lambda_i(x) \text{Hess } \gamma(e_i, e_i)$$

we have then

$$\begin{aligned}
 L\gamma &\leq \text{Tr } P \cdot \ln \left(\int_0^\gamma \frac{ds}{\sqrt{G(s)}} + 1 \right) \sqrt{G(\gamma)} \left(\int_0^\gamma \frac{ds}{\sqrt{G(s)}} + 1 \right) \left(3 + \frac{1}{\rho} \right) + \frac{1}{\rho} |V| \sqrt{G(\gamma)} \left(\int_0^\gamma \frac{ds}{\sqrt{G(s)}} + 1 \right) \\
 &\leq \sqrt{G(\gamma)} \left(\int_0^\gamma \frac{ds}{\sqrt{G(s)}} + 1 \right) \left(3 + \frac{D}{\rho} \right) \prod_{j=1}^k \ln^{(j)} \left(\int_0^\gamma \frac{ds}{\sqrt{G(s)}} + 1 \right)
 \end{aligned}$$

By Theorem 1 and Remark 1 the Omori-Yau maximum principle holds on M . \square

4. CURVATURE ESTIMATES

4.1. The operators L_r . Let $f: M \hookrightarrow N$ be an isometric immersion of a connected n -dimensional Riemannian manifold M into the $(n+1)$ -dimensional Riemannian manifold N . Let $\alpha(X) = -\bar{\nabla}_X \eta$, $X \in TM$ be the second fundamental form of the immersion with respect to a locally defined normal vector field η , where $\bar{\nabla}$ is the Levi-Civita connection of N . Its eigenvalues $\kappa_1, \dots, \kappa_n$ are the principal curvatures of the hypersurface M . The elementary symmetric functions of the principal curvatures are defined by

$$(26) \quad S_0 = 1, \quad S_r = \sum_{i_1 < \dots < i_r} \kappa_{i_1} \cdots \kappa_{i_r}, \quad 1 \leq r \leq n.$$

The elementary symmetric functions S_r define the r -mean curvature H_r of the immersion by

$$(27) \quad H_r = \binom{n}{r}^{-1} S_r,$$

so that H_1 is the mean curvature and H_n is the Gauss-Kronecker curvature. The Newton tensors $P_r: TM \rightarrow TM$, for $r = 0, 1, \dots, n$, are defined setting $P_0 = I$ and $P_r = S_r \text{Id} - \alpha P_{r-1}$ so that

$P_r(x) : T_x M \rightarrow T_x M$ is a self-adjoint linear operator with the same eigenvectors as α . From here we will be following [13, p.3] closely. When r is even the sign of S_r does not depend on η which implies that the tensor P_r is globally defined on TM . If r is odd we will assume that M is two sided, i.e. there exists a globally defined unit normal vector field η in $f(M)$. When a hypersurface is two sided, a choice of η makes P_r globally defined. To give an uniform treatment in what follows we shall assume from now on that M is two-sided.

For each $u \in C^2(M)$, define a symmetric operator $\text{hess } u : TM \rightarrow TM$ by

$$(28) \quad \text{hess } u(X) = \nabla_X \text{grad } u$$

for every $X \in TM$, where ∇ be the Levi-Civita connection of M and the symmetric bilinear form $\text{Hess } u : TM \times TM \rightarrow C^\infty(M)$ by

$$(29) \quad \text{Hess } u(X, Y) = \langle \text{hess } u(X), Y \rangle.$$

Associated to each Newton operator $P_r : TM \rightarrow TM$ there is a second order self-adjoint differential operator $L_r : C^\infty(M) \rightarrow C^\infty(M)$ defined by

$$(30) \quad L_r(u) = \text{Tr}(P_r \text{hess } u) = \text{div}(P_r \text{grad } u) - \langle \text{Tr}(\nabla P_r), \text{grad } u \rangle$$

However, these operators may be not elliptic. Regarding the ellipticity of the L_r 's, one sees that the operator L_r is elliptic if and only if P_r is positive definite. There are geometric conditions implying the positiveness of the P_r and thus the ellipticity of the L_r , e.g. $H_2 > 0$ implies that $H_1 > 0$ by the well known inequality $H_1^2 \geq H_2$. And that implies that all the eigenvalues of P_1 are positive and the ellipticity of L_1 . For the ellipticity of L_r , $r \geq 2$, it is enough to assume that there exists an elliptic point $p \in M$, i.e. a point where the second fundamental form α is positive definite (with respect to an orientation) and $H_{r+1} > 0$. See details in [14], [19], [23], [39]. In this section we are going to apply Theorem 1 to the operators L_r in order to derive curvature estimates.

Again, we denote by $N^{n+1} = I \times_\rho P^n$ the product manifold $I \times P^n$ endowed with the warped product metric $dt^2 + \rho^2(t)d^2P$, where $I \subset \mathbb{R}$ is a open interval, P^n is a complete Riemannian manifold and $\rho : I \rightarrow \mathbb{R}_+$ is a smooth function. Given an isometrically immersed hypersurface $f : M^n \rightarrow N^{n+1}$, define $h : M^n \rightarrow I$ the $C^\infty(M^n)$ height function by setting $h = \pi_I \circ f$, where $\pi_I : N^{n+1} = I \times_\rho P^n \rightarrow I$ is the projection on the first factor. We will need the following result similar to [13, Prop. 6].

Lemma 2. *Let $f : M^n \rightarrow I \times_\rho P^n = N^{n+1}$ be an isometric immersion into a warped product space. Let h be the height function and define*

$$(31) \quad \sigma(t) = \int_{t_0}^t \rho(s) ds.$$

Then

$$(32) \quad \hat{L}_k \sigma(h) = c_k \rho(h) \left(\mathcal{H}(h) + \Theta \frac{H_{k+1}}{H_k} \right),$$

where $c_k = (n-k) \binom{n}{k}$, $\Theta = \langle \eta, T \rangle$ is the angle function, $\mathcal{H}(h) = \frac{\rho'(h)}{\rho(h)}$ and $\hat{L}_k = \text{Tr}(\hat{P}_k \circ \text{hess})$

with $\hat{P}_k = \frac{P_k}{H_k}$.

Proof. We observe that $\text{grad } \sigma(\mathbf{h}) = \rho(\mathbf{h})\text{grad } \mathbf{h}$ and consequently

$$\begin{aligned}
 \text{hess } \sigma(\mathbf{h})(X) &= \nabla_X \text{grad } \sigma(\mathbf{h}) \\
 (33) \qquad &= \rho(\mathbf{h})\nabla_X \text{grad } \mathbf{h} + \langle \text{grad } (\rho \circ \mathbf{h}), X \rangle \text{grad } \mathbf{h} \\
 &= \rho(\mathbf{h}) \text{hess } \mathbf{h}(X) + \rho'(\mathbf{h})\langle \text{grad } \mathbf{h}, X \rangle \text{grad } \mathbf{h},
 \end{aligned}$$

for all $X \in TM^n$. Therefore,

$$\begin{aligned}
 \hat{L}_k \sigma(\mathbf{h}) &= \text{Tr}(\hat{P}_k \circ \text{hess } \sigma(\mathbf{h})) \\
 &= \sum_i^n \left\langle \frac{1}{H_k} P_k \circ \text{hess } \sigma(\mathbf{h})(e_i), e_i \right\rangle \\
 (34) \qquad &= \frac{1}{H_k} \sum_i^n \left\langle \rho(\mathbf{h}) P_k \circ \text{hess } \mathbf{h}(e_i) + \rho'(\mathbf{h})\langle \text{grad } \mathbf{h}, e_i \rangle P_k \text{grad } \mathbf{h}, e_i \right\rangle \\
 &= \frac{1}{H_k} \left(\rho(\mathbf{h}) \sum_i^n \langle P_k \circ \text{hess } \mathbf{h}(e_i), e_i \rangle + \rho'(\mathbf{h})\langle \text{grad } \mathbf{h}, P_k \text{grad } \mathbf{h} \rangle \right).
 \end{aligned}$$

For the other hand, the gradient of $\pi_I \in C^\infty(M)$ is $\text{grad }^N \pi_I = T$, where T stands for the lifting of $\partial/\partial t \in TI$ to the product $I \times_\rho P^n$. Then,

$$(35) \qquad \text{grad } \mathbf{h} = (\text{grad }^N \pi_I)^\perp = T - \Theta\eta.$$

Since the Levi-Civita connection of a warped product satisfies

$$\bar{\nabla}_X T = \mathcal{H}(X - \langle X, T \rangle T), \quad \forall X \in TN^{n+1}$$

we have

$$\bar{\nabla}_X \text{grad } \mathbf{h} = \mathcal{H}(\mathbf{h})(X - \langle X, T \rangle T) - X(\Theta)\eta + \Theta AX, \quad \forall X \in TM^n$$

and thus

$$(36) \qquad \text{hess } \mathbf{h}(X) = \mathcal{H}(\mathbf{h})(X - \langle X, \text{grad } \mathbf{h} \rangle \text{grad } \mathbf{h}) + \Theta AX$$

Taking (36) in account in (34) we get

$$\begin{aligned}
 \hat{L}_k \sigma(\mathbf{h}) &= \frac{1}{H_k} [\rho(\mathbf{h}) (\mathcal{H}(\mathbf{h})(\text{Tr } P_k - \langle P_k \text{grad } \mathbf{h}, \text{grad } \mathbf{h} \rangle) + \Theta \text{Tr}(P_k A)) + \rho'(\mathbf{h})\langle \text{grad } \mathbf{h}, P_k \text{grad } \mathbf{h} \rangle] \\
 &= \frac{1}{H_k} [\rho(\mathbf{h}) (\mathcal{H}(\mathbf{h})(c_k H_k - \langle P_k \text{grad } \mathbf{h}, \text{grad } \mathbf{h} \rangle) + c_k \Theta H_{k+1}) + \rho'(\mathbf{h})\langle \text{grad } \mathbf{h}, P_k \text{grad } \mathbf{h} \rangle] \\
 &= \frac{1}{H_k} [\rho'(\mathbf{h})c_k H_k - \rho'(\mathbf{h})\langle P_k \text{grad } \mathbf{h}, \text{grad } \mathbf{h} \rangle + \rho(\mathbf{h})c_k \Theta H_{k+1} + \rho'(\mathbf{h})\langle P_k \text{grad } \mathbf{h}, \text{grad } \mathbf{h} \rangle] \\
 &= c_k \rho(\mathbf{h}) \left(\mathcal{H}(\mathbf{h}) + \Theta \frac{H_{k+1}}{H_k} \right)
 \end{aligned}$$

□

The following result, (Theorem 3), generalizes [13, Thm.10]. We will assume that

$$(37) \qquad \lim_{x \rightarrow \infty} \frac{\sigma \circ \mathbf{h}(x)}{\varphi(\gamma(x))} = 0$$

where σ is given in (31) and φ is given by

$$(38) \quad \varphi(t) = \ln \left(\int_0^t \frac{ds}{\sqrt{G(s)}} + 1 \right)$$

while (G, γ) is an Omori-Yau pair in the sense of Definition 2.

Theorem 3. *Let $f: M^n \rightarrow I \times_\rho P^n = N^{n+1}$ be a properly immersed hypersurface with second fundamental form α satisfying (17) and $H_2 > 0$. Suppose that P^n carry an Omori-Yau pair (G, γ) for the Hessian, that $\inf \rho > 0$ and $\mathcal{H} = \frac{\rho'}{\rho}$ satisfies (16). If $\text{Tr}(\hat{P}_1) \leq \left(\int_0^{\gamma \circ \pi_P(f)} \frac{ds}{\sqrt{G(s)}} + 1 \right)$ and if the height function $\sigma \circ h$ satisfies (37) then*

$$(39) \quad \sup_M H_2^{\frac{1}{2}} \geq \inf_M \mathcal{H}(h).$$

Proof. By Theorem 2 the Omori-Yau maximum principle for \hat{L}_1 holds on M and then there exists a sequence $x_j \in M$ such that

$$(40) \quad \begin{aligned} \frac{1}{j} &> \hat{L}_1 \sigma \circ h(x_j) = n(n-1)\rho(h(x_j)) \left(\mathcal{H}(h(x_j)) + \Theta(x_j) \frac{H_2}{H_1}(x_j) \right) \\ &\geq n(n-1)\rho(h(x_j)) \left(\mathcal{H}(h(x_j)) - \frac{H_2}{H_1}(x_j) \right) \\ &\geq n(n-1)\rho(h(x_j)) \left(\mathcal{H}(h(x_j)) - \sqrt{H_2}(x_j) \right) \\ &\geq n(n-1)\rho(h(x_j)) \left(\inf_M \mathcal{H}(h) - \sup_M \sqrt{H_2} \right). \end{aligned}$$

Where we used Lemma 2 in the right hand side the first line of (40). Since σ is strictly increasing and $h(x_j) \rightarrow \sup h$ as $j \rightarrow +\infty$, we obtain that

$$\sup_M \sqrt{H_2} \geq \inf_M \mathcal{H}(h).$$

□

Corollary 2. *Let P^n be a complete, non-compact Riemannian manifold whose radial sectional curvature satisfies*

$$(41) \quad K_P^{\text{rad}} \geq -C \cdot G(r),$$

where $G \in C^\infty([0, +\infty))$ is even at the origin and satisfies iv) in Theorem 1, $r(x) = \text{dist}_M(x_0, x)$. If $f: M^n \rightarrow I \times_\rho P^n = N^{n+1}$ is a properly immersed hypersurface with $H_2 > 0$, satisfying (16), (17),

$$(37) \text{ and } \inf \rho > 0. \text{ If } \text{Tr}(\hat{P}_1) \leq \left(\int_0^{r \circ \pi_P(f)} \frac{ds}{\sqrt{G(s)}} + 1 \right) \text{ then}$$

$$(42) \quad \sup_M H_2^{\frac{1}{2}} \geq \inf_M \mathcal{H}(h).$$

Our next result is just a version of Theorem 3 for higher order mean curvatures.

Theorem 4. *Let $f: M^n \rightarrow I \times_\rho P^n = N^{n+1}$ be a proper isometric immersion with an elliptic point and $H_k > 0$. Suppose that the second fundamental form α satisfies (17) and that $\inf \rho > 0$. Moreover, assume that $\mathcal{H} = \rho'/\rho$ satisfies (16) and that P^n carry an Omori-Yau pair (G, γ) for the*

Hessian. If $\text{Tr}(\hat{P}_{k-1}) \leq \left(\int_0^{Y \circ \pi_P(f)} \frac{ds}{\sqrt{G(s)}} + 1 \right)$, $3 \leq k \leq n$ and if the height function $\sigma \circ h$ satisfies (37) then

$$(43) \quad \sup_M H_k^{\frac{1}{k}} \geq \inf_M \mathcal{H}(h).$$

Likewise, we have the corollary

Corollary 3. *Let P^n be a complete, non-compact Riemannian manifold whose radial sectional curvature satisfies*

$$(44) \quad K_P^{\text{rad}} \geq -C \cdot G(r)$$

If $f: M^n \rightarrow I \times_\rho P^n = N^{n+1}$ is a properly immersed hypersurface with $H_k > 0$, satisfying (16), (17), (37) and $\inf \rho > 0$. If $\text{Tr}(\hat{P}_1) \leq \left(\int_0^{r \circ \pi_P(f)} \frac{ds}{\sqrt{G(s)}} + 1 \right)$ then

$$(45) \quad \sup_M H_k^{\frac{1}{k}} \geq \inf_M \mathcal{H}(h).$$

5. SLICE THEOREM

In this section we will prove an extension of the Slice Theorem proved by L. Alias, D. Impera and M. Rigoli [13, Thms. 16 & 21]. We start with the following lemma.

Lemma 3. *Let $f: M^n \rightarrow I \times_\rho P^n = N^{n+1}$ be a hypersurface with non-vanishing mean curvature and suppose that the height function h satisfies*

$$(46) \quad \lim_{x \rightarrow \infty} \frac{h(x)}{\varphi(\gamma(x))} = 0,$$

where φ is given in (6) and γ in the statement of Theorem 1. Assume that $\mathcal{H}' \geq 0$ and that the angle function Θ does not change sign. Choose on M^n the orientation so that $H_1 > 0$. Suppose the Omori-Yau maximum principle for the Laplacian holds on M^n . Then we have that

i) If $\Theta \leq 0$ then $\mathcal{H}(h) \geq 0$,

ii) If $\Theta \geq 0$ then $\mathcal{H}(h) \leq 0$.

Proof. By hypothesis, we have that Omori-Yau maximum principle for the Laplacian holds for the height function h , therefore, there exists a sequences $\{x_j\}, \{y_j\} \subset M^n$ such that

$$(47) \quad \lim_{j \rightarrow +\infty} h(x_j) = \sup h = h^*, \quad |\text{grad } h|^2(x_j) < \left(\frac{1}{j}\right)^2 \quad \text{and} \quad \Delta h(x_j) < \frac{1}{j}$$

and

$$(48) \quad \lim_{j \rightarrow +\infty} h(y_j) = \inf h = h_*, \quad |\text{grad } h|^2(y_j) < \left(\frac{1}{j}\right)^2 \quad \text{and} \quad \Delta h(y_j) > -\frac{1}{j}.$$

On the other hand, we know that $\Delta h = \mathcal{H}(h)(n - |\text{grad } h|^2) + nH_1\Theta$. Therefore, supposing that $\Theta \geq 0$ we get by (47) that

$$0 \geq -nH_1(x_j)\Theta(x_j) > -\frac{1}{j} + \mathcal{H}(h(x_j))(n - |\text{grad } h|^2(x_j))$$

and then, for $j \gg 1$ we obtain

$$0 \geq \frac{-nH_1(x_j)\Theta(x_j)}{n - |\text{grad } h|^2(x_j)} > -\frac{1}{j(n - |\text{grad } h|^2(x_j))} + \mathcal{H}(h(x_j)).$$

letting $j \rightarrow +\infty$, we have

$$(49) \quad 0 \geq \limsup_{j \rightarrow +\infty} \mathcal{H}(h(x_j)) = \mathcal{H}(h^*) \geq \mathcal{H}(h).$$

Similar proof gives the item i). \square

Define the operator $\mathcal{L}_1 = \text{Tr}(\mathcal{P}_1 \circ \text{hess}) = (n-1)\mathcal{H}(h)\Delta - \Theta L_1$ where $\mathcal{P}_1 = (n-1)\mathcal{H}(h)I - \Theta P_1$. Using that

$$\Delta h = \mathcal{H}(h)(n - |\text{grad } h|^2) + nH_1\Theta$$

and

$$L_1(h) = n(n-1)(\mathcal{H}(h)H_1 + \Theta H_2) - \mathcal{H}(h)\langle P_1 \text{grad } h, \text{grad } h \rangle$$

we obtain

$$\begin{aligned} \mathcal{L}_1(h) &= n(n-1)(\mathcal{H}^2(h) - \Theta^2 H_2) - (n-1)\mathcal{H}(h)\langle P_1 \text{grad } h, \text{grad } h \rangle \\ \mathcal{L}_1(\sigma \circ h) &= n(n-1)\rho(h)(\mathcal{H}^2(h) - \Theta^2 H_2) \end{aligned}$$

The following theorem extends [13, Thm. 16] which extends [9, Thm 2.9].

Theorem 5. *Let $f : M^n \rightarrow I \times_{\rho} P^n = N^{n+1}$ be a complete hypersurface of constant positive 2-mean curvature $H_2 > 0$ with radial sectional curvature K_M^{rad} satisfying*

$$(50) \quad K_M^{\text{rad}} \geq -B^2 \prod_{j=1}^{\ell} \left[\ln^{(j)} \left(\int_0^r \frac{ds}{\sqrt{G(s)}} + 1 \right) + 1 \right]^2 G(r), \text{ for } r(x) \gg 1,$$

where $G \in C^\infty([0, +\infty))$ is even at the origin and satisfies iv) in Theorem 1, $r(x) = \text{dist}_M(x_0, x)$ and $B \in \mathbb{R}$. Suppose also that height function satisfies the conditions (37), (46) and that

$$(51) \quad |H_1|(r) \leq \frac{1}{n(n-1)} \left(\int_0^r \frac{ds}{\sqrt{G(s)}} + 1 \right),$$

and

$$(52) \quad |\mathcal{H}|(t) \leq \frac{1}{n(n-1)} \left(\int_0^t \frac{ds}{\sqrt{G(s)}} + 1 \right)$$

If $\mathcal{H}' > 0$ almost everywhere and the angle function Θ does not change sign, then $f(M^n)$ is a slice.

Proof. Taking an orientation on M^n in which $H_1 > 0$ we have, by Lemma 3, that if $\Theta \leq 0$, then $\mathcal{H}(h) \geq 0$ and therefore the operator \mathcal{P}_1 is positive semi-definite. Therefore, which implies that \mathcal{L}_1 is semi-elliptic. Furthermore, by (61) and (52), we have

$$\text{Tr } \mathcal{P}_1 = n(n-1)(\mathcal{H}(h) - H_1\Theta) \leq \left(\int_0^\gamma \frac{ds}{\sqrt{G(s)}} + 1 \right)$$

By Corollary 1 the Omori-Yau maximum principle for the operator \mathcal{L}_1 holds on M^n with the functions h and $\sigma \circ h$ (conditions (37) and (46)). Thus, there is a sequence $\{x_j\} \subset M^n$ such that

- i. $\lim_{j \rightarrow +\infty} \sigma(h(x_j)) = (\sigma \circ h)^*$
- ii. $|\text{grad } (\sigma \circ h)|(x_j) = \rho(h(x_j))|\text{grad } h|(x_j) < \frac{1}{j}$

$$\text{iii. } \mathcal{L}_1(\sigma \circ h)(x_j) < \frac{1}{j}.$$

We know that $\mathcal{H}(h) \geq 0$, and hence $\rho'(h) \geq 0$ and ρ is increasing. Since σ is strictly increasing, we have $\lim_{j \rightarrow +\infty} h(x_j) = h^* \leq \infty$. Thus, $\lim_{j \rightarrow +\infty} \rho(h(x_j)) > c > 0$. Since $\Theta^2 \leq 1$, the item iii. tells us that

$$\begin{aligned} \frac{1}{j} &> = n(n-1)\rho(h(x_j)) (\mathcal{H}^2(h(x_j)) - \Theta^2(x_j)H_2) \\ &> n(n-1)\rho(h(x_j)) (\mathcal{H}^2(h(x_j)) - H_2). \end{aligned}$$

Making $j \rightarrow +\infty$, we gets

$$(53) \quad \mathcal{H}^2(h^*) \leq H_2.$$

On the other hand, there is a sequence $\{y_j\} \subset M^n$, in such a way that

- i. $\lim_{j \rightarrow +\infty} h(y_j) = h_*$
- ii. $|\text{grad } h|(y_j) = |\text{grad } h|(y_j) < \frac{1}{j}$
- iii. $\mathcal{L}_1(h)(y_j) > -\frac{1}{j}$.

Since \mathcal{P}_1 is positive semi-definite, there exists a constant $\beta \geq 0$ such that

$$\langle \mathcal{P}_1 \text{grad } h, \text{grad } h \rangle \geq \beta |\text{grad } h|^2.$$

Thus,

$$\begin{aligned} -\frac{1}{j} &< n(n-1) (\mathcal{H}^2(h(y_j)) - \Theta^2(y_j)H_2(y_j)) - \mathcal{H}(h(y_j)) \langle \mathcal{P}_1 \text{grad } h(y_j), \text{grad } h(y_j) \rangle \\ &\leq n(n-1) (\mathcal{H}^2(h(y_j)) - \Theta^2(y_j)H_2(y_j)) - \beta |\text{grad } h|^2(y_j) \\ &\leq n(n-1) (\mathcal{H}^2(h(y_j)) - \Theta^2(y_j)H_2(y_j)). \end{aligned}$$

Since $1 \geq \Theta^2(y_j) = 1 - |\text{grad } h|^2(y_j) > 1 - \frac{1}{j}$, by the item ii. we have $\lim_{j \rightarrow +\infty} \Theta^2(y_j) = 1$. Doing $j \rightarrow +\infty$, we get

$$(54) \quad H_2 \leq \mathcal{H}^2(h_*).$$

Combining (53) with (54), we get that

$$\mathcal{H}(h^*) \leq H_2 \leq \mathcal{H}(h_*)$$

and as \mathcal{H} is an increasing function, conclude that $h^* = h_* < \infty$.

If $\Theta \geq 0$, we applied in a manner entirely analogous the Omori-Yau maximum principle to the semi-elliptic operator $-\mathcal{L}_1$. \square

To extend the previous findings in the case of higher order curvatures, let us define for each $2 \leq k \leq n$, the operators

$$(55) \quad \mathcal{L}_{k-1} = \text{Tr}(\mathcal{P}_{k-1} \circ \text{hess}),$$

where

$$(56) \quad \mathcal{P}_{k-1} = \sum_{j=0}^{k-1} (-1)^j \frac{c_{k-1}}{c_j} \mathcal{H}^{k-1-j}(h) \Theta^j \mathcal{P}_j.$$

Observe that

$$\mathcal{P}_{k-1} = \frac{c_{k-1}}{c_{k-2}} \mathcal{H}(h) \mathcal{P}_{k-2} + (-1)^{k-1} \Theta^{k-1} \mathcal{P}_{k-1}$$

and consequently

$$(57) \quad \mathcal{L}_{k-1} = \frac{c_{k-1}}{c_{k-2}} \mathcal{H}(h) \mathcal{L}_{k-2} + (-1)^{k-1} \Theta^{k-1} \mathcal{L}_{k-1}.$$

By induction, we see that

$$(58) \quad \mathcal{L}_{k-1} h = c_{k-1} (\mathcal{H}^k(h) - (-1)^k \Theta^k H_k) - \mathcal{H}(h) \langle \mathcal{P}_{k-1} \text{grad } h, \text{grad } h \rangle$$

and

$$(59) \quad \mathcal{L}_{k-1} \sigma(h) = c_{k-1} \rho(h) (\mathcal{H}^k(h) - (-1)^k \Theta^k H_k).$$

In fact, assuming that the expression (58) is valid for $k-2$, we have:

$$\begin{aligned} \mathcal{L}_{k-1} h &= \frac{c_{k-1}}{c_{k-2}} \mathcal{H}(h) \mathcal{L}_{k-2} h + (-1)^{k-1} \Theta^{k-1} \mathcal{L}_{k-1} h \\ &= \frac{c_{k-1}}{c_{k-2}} \mathcal{H}(h) [c_{k-2} (\mathcal{H}^{k-1}(h) - (-1)^{k-1} \Theta^{k-1} H_{k-1}) - \mathcal{H}(h) \langle \mathcal{P}_{k-2} \text{grad } h, \text{grad } h \rangle] + \\ &\quad + (-1)^{k-1} \Theta^{k-1} [\mathcal{H}(h) (c_{k-1} H_{k-1} - \langle \mathcal{P}_{k-1} \text{grad } h, \text{grad } h \rangle) + c_{k-1} \Theta H_k] \\ &= c_{k-1} \mathcal{H}^k(h) - (-1)^{k-1} c_{k-1} \mathcal{H}(h) \Theta^{k-1} H_{k-1} - \frac{c_{k-1}}{c_{k-2}} \mathcal{H}^2(h) \langle \mathcal{P}_{k-2} \text{grad } h, \text{grad } h \rangle + \\ &\quad + (-1)^{k-1} c_{k-1} \mathcal{H}(h) \Theta^{k-1} H_{k-1} + (-1)^{k-1} c_{k-1} \Theta^k H_k - \\ &\quad - (-1)^{k-1} \mathcal{H}(h) \Theta^{k-1} \langle \mathcal{P}_{k-1} \text{grad } h, \text{grad } h \rangle \\ &= c_{k-1} (\mathcal{H}^k(h) - (-1)^k \Theta^k H_k) - \mathcal{H}(h) \langle \mathcal{P}_{k-1} \text{grad } h, \text{grad } h \rangle. \end{aligned}$$

The proof of the expression (59) follows similarly.

The next theorem extends the Theorem 5 for the case of higher order curvatures and your proof is analogous and use only the expressions (58) and (59).

Theorem 6. *Let $f : M^n \rightarrow I \times_{\rho} P^n = N^{n+1}$ be a complete hypersurface of constant positive k -mean curvature $H_k > 0$, $3 \leq k \leq n$ with radial sectional curvature K_M^{rad} satisfying*

$$(60) \quad K_M^{\text{rad}} \geq -B^2 \prod_{j=1}^{\ell} \left[\ln^{(j)} \left(\int_0^r \frac{ds}{\sqrt{G(s)}} + 1 \right) + 1 \right]^2 G(r), \quad \text{for } r(x) \gg 1,$$

where $G \in C^\infty([0, +\infty))$ is even at the origin and satisfies iv) in Theorem 1, $r(x) = \text{dist}_M(x_0, x)$ and $B \in \mathbb{R}$. Suppose also that height function satisfies the conditions (37), (46) and that

$$(61) \quad |H_1^j| \cdot |\mathcal{H}^{k-1-j}(h)|(r) \leq \frac{1}{kc_{k-1}} \left(\int_0^r \frac{ds}{\sqrt{G(s)}} + 1 \right), \quad \forall j = 0, \dots, k-1.$$

Assume that there exists an elliptic point in M^n . If $\mathcal{H}' > 0$ almost everywhere and the angle function Θ does not change sign, then $f(M^n)$ is a slice.

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